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Short Communication

A note on the frequency of nonlinear conservative oscillators

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1. Introduction

Among the topics in nonlinear oscillations, free vibration of an oscillator with a nonlinear spring is one that has attracted considerable interest. In the study of such a nonlinear system, approximate solutions, rather than the exact one, are often sought because the latter is possible only for a relatively few nonlinear systems. Consider the following nonlinear system:

$$\ddot{x} + \sum_{i=1}^{n} f_i(x) = 0, \quad x(0) = A, \quad \dot{x}(0) = 0, \tag{1}$$

where overdots denote differentiations with respect to time t and $f_i(-x) = -f_i(x)$. The conservative nonlinear system (1) may be very complex. Therefore, we first consider the following auxiliary equations:

$$\ddot{x} + f_i(x) = 0, \quad x(0) = A, \ \dot{x}(0) = 0, \ i = 1, 2, \ldots, n. \tag{2}$$

It is assumed that ω_i $(i = 1, 2, \ldots, n)$ are the approximations to the true natural frequencies ω_{ei} $(i = 1, 2, \ldots, n)$ of the nonlinear oscillators modeled by Eqs. (2), and ω is the approximation to the true natural frequency ω_e of the nonlinear oscillator described by Eq. (1).

Natural frequency analysis is useful for the investigation of stability, bifurcation, resonance and chaos in nonlinear dynamic systems. Now an interesting question arises: What is the relation between ω and ω_i $(i = 1, 2, \ldots, n)$? The main purpose of this paper is to show that the following

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approximate relation holds:

$$\omega^2 = \sum_{i=1}^{n} \omega_i^2. \tag{3}$$

The paper then gives two examples to illustrate the applications of Eq. (3). Since any one of Eqs. (2) is simpler than Eq. (1), some of them may have exact solutions. These two examples will show that if ω_i $(i = 1, 2, \ldots, n)$ in Eq. (3) are replaced by ω_{ei} $(i = 1, 2, \ldots, n)$, i.e.,

$$\omega^2 = \sum_{i=1}^{n} \omega_{ei}^2, \tag{4}$$

then formula (4) can give good approximate frequencies.

2. Proof of Eq. (3)

Many perturbation techniques exist for constructing analytical approximations to the oscillatory solution of second order, nonlinear differential equations (1) and (2) [1,2]. These methods are, in principle, for solving problems with small parameter. But the method of harmonic balance can be applied to nonlinear oscillatory problems for which the nonlinear terms are not "small" [1,3]. Therefore, the approximate analytical solutions to Eqs. (1) and (2) can be obtained by using this method.

To proceed, we introduce the transformations $\tau_i = \omega_i t$ $(i = 1, 2, \ldots, n)$. Hence, Eqs. (2) become

$$\omega_i^2 x'' + f_i(x) = 0, \quad x(0) = A, \;\; x'(0) = 0, \;\; i = 1, 2, \ldots, n, \tag{5}$$

where the primes indicate the derivatives with respect to τ_i. The first approximations to Eqs. (5) are taken to be

$$x(\tau_i) = A\cos\tau_i, \quad i = 1, 2, \ldots, n. \tag{6}$$

Substitution of Eq. (6) into Eqs. (5) gives

$$(-A\omega_i^2 + a_i)\cos\tau_i + (\text{higher-order harmonics}) = 0, \quad i = 1, 2, \ldots, n, \tag{7}$$

where

$$a_i = \frac{2}{\pi}\int_0^{\pi} f_i(A\cos\tau_i)\cos\tau_i \,\mathrm{d}\tau_i, \quad i = 1, 2, \ldots, n. \tag{8}$$

It is assumed that $a_i > 0$ $(i = 1, 2, \ldots, n)$. Setting the coefficient of $\cos\tau$ equal to zero yields

$$a_i = A\omega_i^2, \quad i = 1, 2, \ldots, n. \tag{9}$$

Introducing the substitution $\tau = \omega t$ into Eq. (1) results in

$$\omega^2 x'' + \sum_{i=1}^{n} f_i(x) = 0, \quad x(0) = A, \quad x'(0) = 0, \tag{10}$$

where the primes indicate the derivatives with respect to τ. The first approximation to Eq. (10) is assumed to be

$$x(\tau) = A\cos\tau. \tag{11}$$

Substituting Eq. (11) into Eq. (10) gives

$$\left(-\omega^2 A + \sum_{i=1}^{n} a_i\right)\cos\tau + (\text{higher-order harmonics}) = 0. \tag{12}$$

Obviously,

$$a_i = \frac{2}{\pi}\int_0^{\pi} f_i(A\cos\tau)\cos\tau\,\mathrm{d}\tau = \frac{2}{\pi}\int_0^{\pi} f_i(A\cos\tau_i)\cos\tau_i\,\mathrm{d}\tau_i, \quad i = 1, 2, \ldots, n. \tag{13}$$

Substituting Eq. (9) into Eq. (12), setting the coefficient of $\cos\tau$ equal to zero and solving for ω^2 gives Eq. (3) at once.

3. Examples

Example 1. Consider the Duffing equation

$$\ddot{x} + x + \varepsilon x^3 = 0, \quad x(0) = A, \quad \dot{x}(0) = 0, \tag{14}$$

where $\varepsilon \geqslant 0$. We first consider the following two equations:

$$\ddot{x} + x = 0, \quad x(0) = A, \ \dot{x}(0) = 0, \tag{15}$$

$$\ddot{x} + \varepsilon x^3 = 0, \quad x(0) = A, \ \dot{x}(0) = 0. \tag{16}$$

The approximate frequency of Eq. (16) obtained by using the method of harmonic balance is [1]

$$\omega_2\sqrt{\frac{3\varepsilon}{4}}A. \tag{17}$$

Since $\omega_{e1} = 1$, from Eq. (3) the approximate frequency of Eq. (14) can be written as

$$\omega_{a1} = \sqrt{1 + \frac{3}{4}\varepsilon A^2}. \tag{18}$$

In order to improve the accuracy of this formula, we note that the exact frequency of the periodic motion of Eq. (16) is

$$\omega_{e2} = 0.847215\sqrt{\varepsilon}A. \tag{19}$$

The computation of Eq. (19) is given in detail in Appendix A. Then, from Eq. (4) we have the following approximate frequency of Eq. (14):

$$\omega_{a2} = \sqrt{1 + 0.717773\varepsilon A^2}. \tag{20}$$

The exact frequency of the Duffing equation is [4]

$$\omega_e = \frac{\pi\sqrt{1+\varepsilon A^2}}{2}\left(\int_0^{\pi/2}\frac{\mathrm{d}\theta}{\sqrt{1-m^2\sin^2\theta}}\right)^{-1}, \quad m = \frac{\varepsilon A^2}{2(1+\varepsilon A^2)}. \tag{21}$$

The corresponding second approximate frequency obtained by using a classical perturbation method [5] is

$$\omega_{a3} = \tfrac{1}{4}\sqrt{8+6\varepsilon A^2+\sqrt{64+96\varepsilon A^2+30\varepsilon^2A^4}}. \tag{22}$$

For comparison, the exact frequency ω_e obtained by integrating Eq. (21) and the approximate frequencies computed by Eqs. (18), (20) and (22), respectively, are listed in Table 1. Table 1 shows that formula (20) is more accurate than formulas (18) and (22) for large values of εA^2. We also have

$$\lim_{\varepsilon A^2\to\infty}\frac{\omega_{a1}}{\omega_e} = \frac{\sqrt{3}}{\pi}\int_0^{\pi/2}\frac{\mathrm{d}\theta}{\sqrt{1-0.5\sin^2\theta}} = \frac{\sqrt{3}}{\pi}\times 1.8541 = 1.0222, \tag{23}$$

$$\lim_{\varepsilon A^2\to\infty}\frac{\omega_{a2}}{\omega_e} = \frac{2\sqrt{0.717773}}{\pi}\int_0^{\pi/2}\frac{\mathrm{d}\theta}{\sqrt{1-0.5\sin^2\theta}} = \frac{2\sqrt{0.717773}}{\pi}\times 1.8541 = 1.0000. \tag{24}$$

Eq. (24) implies that for very large values of εA^2, Eq. (14) may be replaced by Eq. (16).

The exact periodic solution to Eq. (14) is [6]

$$x_e(t) = A\,\mathrm{cn}(wt,k), \tag{25}$$

where $\mathrm{cn}(wt,k)$ is the cosine Jacobian elliptic function,

$$w = \sqrt{1+\varepsilon A^2} \quad \text{and} \quad k = \sqrt{\frac{1}{2}\left(1-\frac{1}{1+\varepsilon A^2}\right)}.$$

Table 1
Comparison of approximate frequencies with the corresponding exact frequency for the Duffing equation

εA^2	ω_e (Eq. (21))	ω_{a1} (Eq. (18))	ω_{a2} (Eq. (20))	ω_{a3} (Eq. (22))
0.2	1.07200	1.07238	1.06937	1.07200
0.4	1.13891	1.14018	1.13451	1.13891
0.6	1.20173	1.20416	1.19610	1.20173
0.8	1.26118	1.26491	1.25468	1.26118
1	1.31778	1.32288	1.31064	1.31776
2	1.56911	1.58114	1.56062	1.56905
5	2.15042	2.17945	2.14216	2.15018
10	2.86664	2.91548	2.85967	2.86613
100	8.53359	8.71780	8.53096	8.53110
1000	26.8107	27.4044	26.8099	26.8025
10 000	84.7275	86.6083	84.7274	84.7013

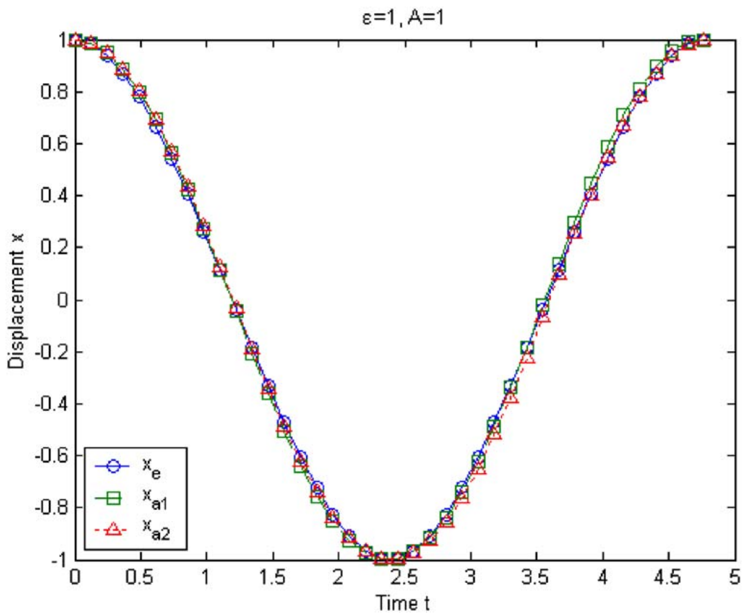

Fig. 1. Comparison of the approximate periodic solutions with the exact solution for $\varepsilon = 1$, $A = 1$.

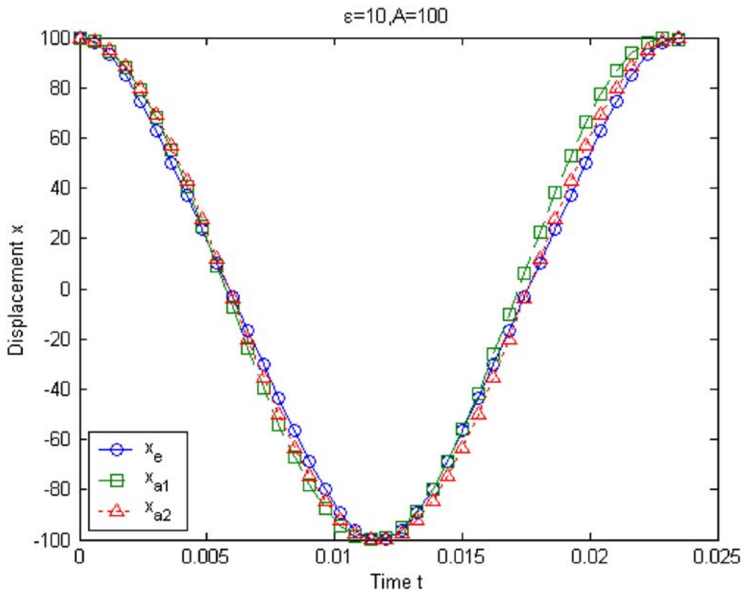

Fig. 2. Comparison of the approximate periodic solutions with the exact solution for $\varepsilon = 10$, $A = 100$.

The approximate periodic solutions $x_{a1}(t)$ and $x_{a2}(t)$ corresponding to Eqs. (18) and (20) are

$$x_{a1}(t) = A\cos\omega_{a1}t \tag{26}$$

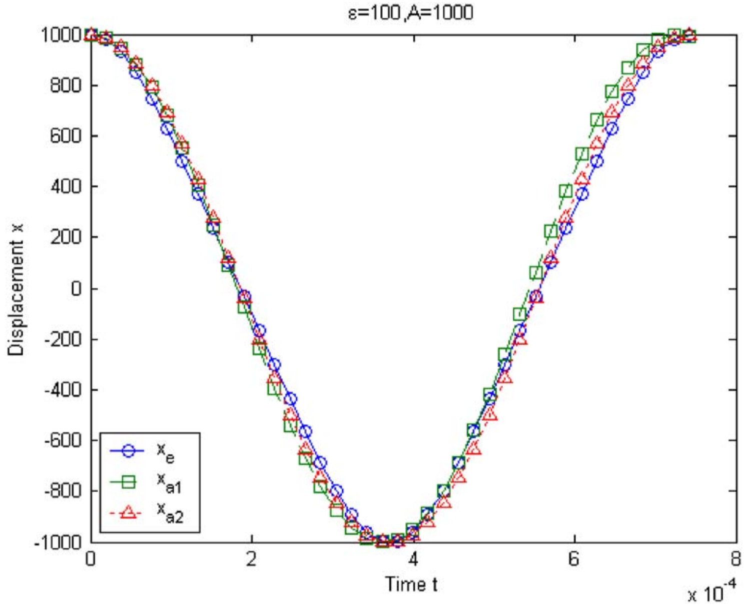

Fig. 3. Comparison of the approximate periodic solutions with the exact solution for $\varepsilon = 100$, $A = 1000$.

and

$$x_{a2}(t) = A \cos \omega_{a2} t, \tag{27}$$

respectively. A comparison of periodic solutions $x_e(t)$, $x_{a1}(t)$ and $x_{a2}(t)$ is presented in Figs. 1–3 for: $\varepsilon = 1$, $A = 1$; $\varepsilon = 10$, $A = 100$; $\varepsilon = 100$, $A = 1000$; respectively. Figs. 1–3 indicate that $x_{a1}(t)$ and $x_{a2}(t)$ are close to $x_e(t)$.

Example 2. Consider the nonlinear differential equation [7]

$$\ddot{x} + x^{1/3} + \varepsilon x^3 = 0, \quad x(0) = A, \;\; \dot{x}(0) = 0, \tag{28}$$

where $\varepsilon \geqslant 0$. The corresponding two auxiliary equations are

$$\ddot{x} + x^{1/3} = 0, \quad x(0) = A, \quad \dot{x}(0) = 0, \tag{29}$$

$$\ddot{x} + \varepsilon x^3 = 0, \quad x(0) = A, \quad \dot{x}(0) = 0. \tag{30}$$

For Eq. (29), the method of harmonic balance gives [8]

$$\omega_1 = \left(\frac{4}{3A^2}\right)^{1/6}. \tag{31}$$

Substituting Eqs. (17) and (32) into Eq. (3), we obtain the following approximate frequency of Eq. (28):

$$\omega_{b1} = \sqrt{\left(\frac{4}{3A^2}\right)^{1/3} + \frac{3}{4}\varepsilon A^2}. \tag{32}$$

Table 2
Comparison of approximate frequencies with the corresponding exact frequency of Eq. (28) for $\varepsilon = 1$

A	ω_e (Eq. (35))	ω_{b1} (Eq. (32))	ω_{b2} (Eq. (34))
0.2	1.83904	1.80230	1.83827
0.4	1.49546	1.46540	1.49182
0.6	1.37529	1.34803	1.36717
0.8	1.35030	1.32559	1.33755
1	1.38139	1.36038	1.36515
2	1.91143	1.92181	1.89550
5	4.28687	4.37338	4.28208
10	8.48810	8.67393	8.48671
100	84.7123	86.6028	84.7218
1000	846.387	866.025	847.215
10 000	8330.71	8660.25	8472.15

The exact frequency of Eq. (29) is [9]

$$\omega_{e1} = \frac{\sqrt{\pi}\Gamma(1/4)}{2\sqrt{6}\Gamma(3/4)A^{1/3}} = \frac{1.07045}{A^{1/3}}. \tag{33}$$

Substituting Eqs. (19) and (33) into Eq. (4) yields another approximate frequency of Eq. (28):

$$\omega_{b2} = \sqrt{\frac{1.14586}{A^{2/3}} + 0.717773\varepsilon A^2}. \tag{34}$$

The exact frequency of Eq. (28) is

$$\omega_e = \frac{\pi}{2\sqrt{2}}\left[\int_0^A \frac{\mathrm{d}x}{\sqrt{(3A^{4/3} + \varepsilon A^4) - (3x^{4/3} + \varepsilon x^4)}}\right]^{-1}. \tag{35}$$

For comparison, the exact frequency ω_e obtained by integrating Eq. (35) and the approximate frequencies computed by Eqs. (32) and (34), respectively, are listed in Table 2 for $\varepsilon = 1$. Table 2 shows that formula (34) is more accurate than formula (32) except when $A = 2$. The relative error of ω_{b2} with respect to ω_e is less than 1.70% even when $A = 10\,000$.

The approximate periodic solutions corresponding to Eqs. (32) and (34) are, respectively,

$$x_{b1}(t) = A\cos\omega_{b1}t, \tag{36}$$

$$x_{b2}(t) = A\cos\omega_{b2}t. \tag{37}$$

Figs. 4–6 show the comparison of the numerical periodic solution $x_{\mathrm{num}}(t)$ of Eq. (28) achieved by using Runge–Kutta (R–K) method and the approximate periodic solutions $x_{b1}(t)$ and $x_{b2}(t)$ for: $\varepsilon = 1$, $A = 1$; $\varepsilon = 10$, $A = 100$; $\varepsilon = 100$, $A = 1000$; respectively. They show that $x_{b1}(t)$ and $x_{b2}(t)$ are close to $x_{\mathrm{num}}(t)$.

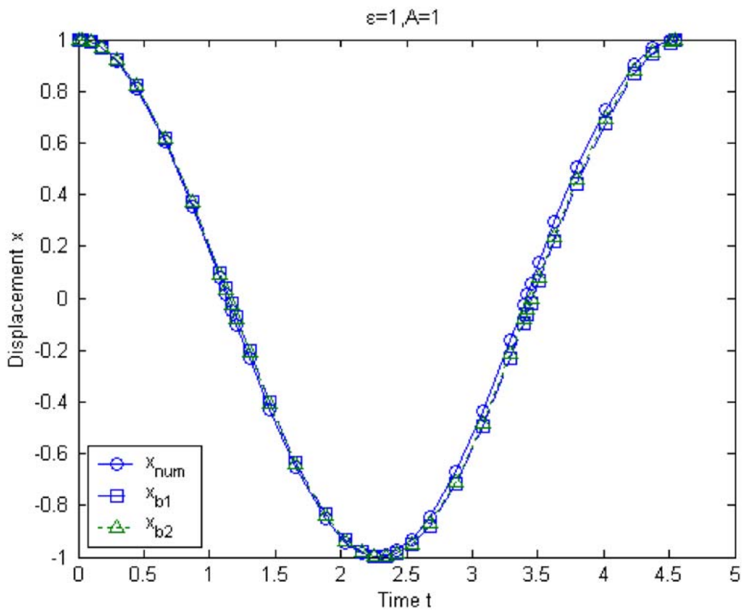

Fig. 4. Comparison of the approximate periodic solutions with the numerical solution for $\varepsilon = 1$, $A = 1$.

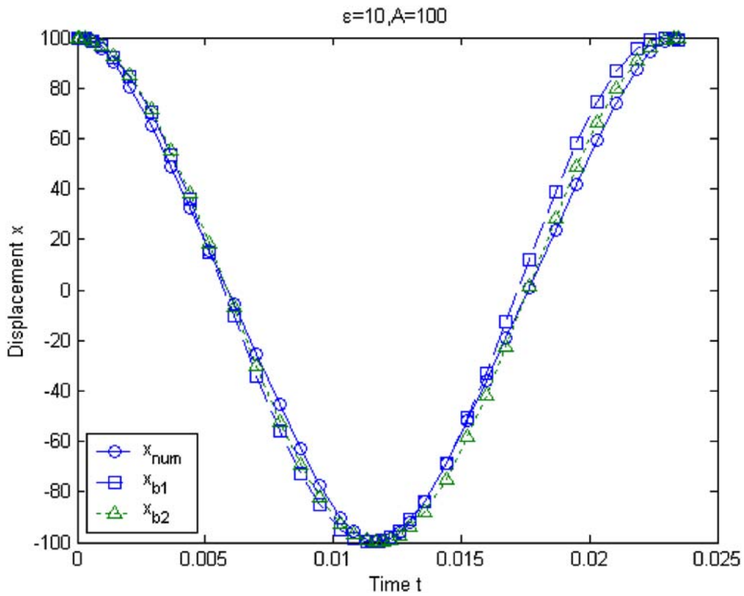

Fig. 5. Comparison of the approximate periodic solutions with the numerical solution for $\varepsilon = 10$, $A = 100$.

4. Concluding remarks

This paper is concerned with the relation between nonlinear conservative systems (1) and (2). Formulas (3) and (4) are the main results. When we meet system (1), we may first consider

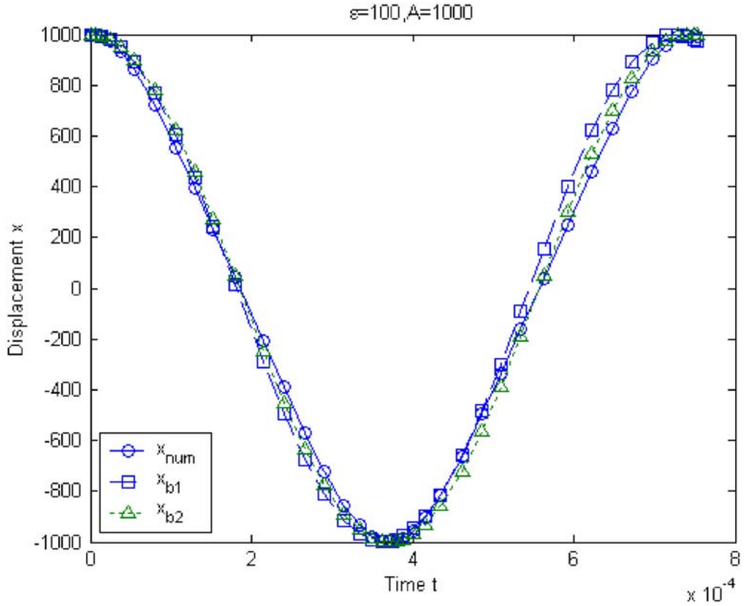

Fig. 6. Comparison of the approximate periodic solutions with the numerical solution for $\varepsilon = 100$, $A = 1000$.

auxiliary systems (2). Two examples have shown that formulas (3) and (4) can provide good approximations. Generally speaking, formula (4) is more accurate than formula (3), especially when the oscillation amplitude A is large. Eq. (18) is the well-known result for the Duffing equation [1,5], and it is consistent with formula (3). In other words, formulas (3) and (4) have been staring us in the face for a long time and we have not noticed. Formula (3) or formula (4) can be regarded as "a superposition principle" for nonlinear conservative systems.

Finally, it should be pointed out that the result is only valid for the vibrations in the small vicinity around the equilibrium position. If there is more than one equilibrium position, the amplitude should be limited in small range. Otherwise, periodic solutions might not exist.

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Appendix A. The computations of ω_{e2}

Eq. (16) can be written as

$$\dot{x}\,\mathrm{d}\dot{x} + \varepsilon x^3\,\mathrm{d}x = 0. \tag{A.1}$$

Integrating of Eq. (A.1), with use made of the initial conditions given in Eq. (16), yields

$$\dot{x} = -\sqrt{\frac{\varepsilon}{2}\,(A^4 - x^4)}, \quad 0 \leqslant t \leqslant T. \tag{A.2}$$

Here, T is the period of the oscillation. Integrating expression (A.2) from t to $t = T/4$, we obtain

$$T = \frac{4\sqrt{2}}{\sqrt{\varepsilon}} \int_0^A \frac{\mathrm{d}x}{\sqrt{A^4 - x^4}} = \frac{4\sqrt{2}}{\sqrt{\varepsilon}A} \int_0^1 \frac{\mathrm{d}y}{\sqrt{1 - y^4}}. \tag{A.3}$$

Letting $y = u^{1/4}$, we have

$$\begin{aligned} T &= \frac{\sqrt{2}}{\sqrt{\varepsilon}A} \int_0^1 \frac{\mathrm{d}u}{u^{3/4}\sqrt{1-u}} = \frac{\sqrt{2}B(1/4, 1/2)}{\sqrt{\varepsilon}A} = \frac{\sqrt{2}\Gamma(1/4)\Gamma(1/2)}{\sqrt{\varepsilon}A\Gamma(3/4)} \\ &= \frac{32}{\pi A\sqrt{\varepsilon}}\,(\Gamma(5/4))^2\Gamma(3/2) = \frac{7.416284}{A\sqrt{\varepsilon}}, \end{aligned} \tag{A.4}$$

where $B(m, n)$ is the Beta function and $\Gamma(n)$ is the Gamma function. Then,

$$\omega_{e2} = \frac{2\pi}{T} = 0.847215\sqrt{\varepsilon}A. \tag{A.5}$$

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